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Universal power-law tails for singularity-dominated strong fluctuations

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Abstract. Integrals of the type $I(t) \equiv \iint \mathrm{d}x\, \mathrm{d}y\, \delta(t-H(x, y))$ are considered, where H is a function from an ensemble labelled by parameters. I can represent circulation times of fluid particles in the plane, orbital periods or semiclassical densities of states for one-dimensional Hamiltonian systems, spectral densities for two-dimensional crystals, or the strength of a wave pulse produced by propagation of a deformed step discontinuity. As H varies over the ensemble, I develops strong fluctuations associated with Legendre singularities. For unrestricted functions H, the probability distribution $P(I)$ describing the fluctuations of I decays according to a universal I^{-9} law, obtained by a scaling argument involving Arnold's classification of catastrophes. If H is only quadratic in y, $P \propto I^{-10}$. If the integral defining I is one-dimensional, $P \propto I^{-3}$; if it is three-dimensional, I decays no faster than I^{-48}. The probability distribution of $I' \equiv |\partial I/\partial t|$ decays as $(I')^{-2}$.

1. Introduction

This paper is intended to demonstrate a remarkable statistical property of integrals I of the type

$$I(t, \{C_i\}) \equiv \iint_{\mathscr{D}} \mathrm{d}x\, \mathrm{d}y\, \delta(t-H(x, y; \{C_i\})). \tag{1}$$

H is an ensemble (i.e. a family) of smooth functions of the two variables x, y labelled by an arbitrary number of parameters $\{C_i\} \equiv \{C_0, C_1 \ldots\}$, and the integration domain $\mathscr{D}$ is the whole xy plane or some fixed finite part of it; all quantities are real. The Dirac δ function implies that I is a property of the contour having height t in the H landscape; this contour need not be simply connected. Obviously I is never negative. Before explaining the problem to be studied, a list will be given illustrating the wide range of physical contexts in which I arises.

(i) *Fluid circulation times.* H is the stream function (vector potential) for a steady two-dimensional incompressible flow, with velocity components

$$v_x \equiv \partial H/\partial y \qquad v_y \equiv -\partial H/\partial x. \tag{2}$$

Then I is the time taken by fluid particles traversing the stream line (contour) within $\mathscr{D}$ and labelled t. To see this, simply denote the arc length along the contour by s, and use (1) (fixing $\{C_i\}$), so that

$$I(t) = \int \frac{\mathrm{d}s}{|\nabla H|} = \int \frac{\mathrm{d}s}{|v|}. \tag{3}$$

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In many cases (whose precise specification is not relevant here) the stream line will consist of one or more closed curves lying entirely inside $\mathscr{D}$.

(ii) *Hamiltonian orbit periods.* H is the Hamiltonian for a system with a single coordinate x and momentum y. The evolution of the system is governed by Hamilton's equations (2), and the same argument as that based on (3) shows that I is the total time taken to traverse all orbits with energy t within $\mathscr{D}$.

(iii) *Semiclassical density of states.* With H as in (ii), I is (apart from a factor (Planck's constant)$^{-1}$) the asymptotic density of eigenstates, at energy t, of the quantal Hamiltonian corresponding to H.

(iv) *Crystal density of states.* x and y are crystal momentum components of electrons in a two-dimensional periodic structure, and H describes a single quantal energy band. Then I is the density of electron states at energy t.

(v) *Wave pulses.* Consider a wave $\phi(X, Y, Z, t)$ in three-dimensional space propagating with speed c according to the ordinary linear scalar non-dispersive wave equation. Initially the wave is a step pulse with a plane wavefront moving in the positive Z direction. For some negative Z values the wave encounters a refracting medium whose effect is to deform the wavefront so that the wave arriving at $Z=0$ is

$$\phi(X, Y, 0, t)=\Theta(t+h(x, y)/c) \tag{4}$$

where Θ is the unit step function and $h(X, Y)$ is the height function describing the wavefront deformation. It is shown in appendix 1 that provided h has small slopes ('paraxiality') the wave reaching $Z>0$ at time t is

$$\phi(X, Y, Z, t)=\frac{1}{2\pi Z}\iint \mathrm{d}x\,\mathrm{d}y\,\delta\Big(ct-\Big\{Z-h(x, y)+\frac{(X-x)^2+(Y-y)^2}{2Z}\Big\}\Big). \tag{5}$$

This is obviously of the form (1), with X, Y and Z considered either as fixed or as three of the parameters $\{C_i\}$ (other parameters will label members of an ensemble of wavefront deformations h).

What will be studied here is the form of the tail of the probability distribution of I over the ensemble parametrised by $\{C_i\}$, i.e.

$$\begin{aligned} P(I)&=\langle\delta(I-I(t;\{C_i\}))\rangle \\ &=\int \mathrm{d}C_0\int \mathrm{d}C_1\ldots P_C(\{C_i\})\delta(I-I(t;\{C_i\})) \qquad \text{as } I\to\infty. \end{aligned} \tag{6}$$

P_C is the density of realisations of the ensemble over $\{C_i\}$; for example if $\{C_i\}$ are the random phases (uniform on $0\leq C_i<2\pi$) in a Gaussian ensemble of functions H, P_C consists simply of a factor $1/2\pi$ for each C_i. The t dependence in (6) turns out to be inconsequential and is not explicitly denoted in $P(I)$.

The tail of $P(I)$ describes *strong fluctuations* in I over the ensemble, and the strong fluctuations arise because functions of type (1) have *singularities* in the space $(t, \{C_i\})$. The singularities arise (cf (3)) whenever ∇H vanishes (i.e. wherever H has a critical point) somewhere on the t contour contributing to (1), and their locus in $(t, \{C_i\})$ space is obtained by eliminating x, y from

$$H(x, y;\{C_i\})=t \qquad \frac{\partial H}{\partial x}(x, y;\{C_i\})=0 \qquad \frac{\partial H}{\partial y}(x, y;\{C_i\})=0. \tag{7}$$

These equations define *Legendre singularities*, whose form and classification are discussed by Arnold (1975, 1976) and Sewell (1977, 1978).

In the examples described above, the Legendre singularities are realised as (i) fluid vortices (i.e. static centres of local rotation) or stagnation points (Berry and Mackley 1977); (ii) and (iii) Hamiltonian elliptic and hyperbolic fixed points (Arnold 1978); (iv) Van Hove singularities (Wannier 1959); (v) geometrical optics wavefronts (Berry 1972, Dangelmayr and Güttinger 1982).

According to (7) the singularities are hypersurfaces with codimension unity in $(t, \{C_i\})$ space. Almost everywhere the surfaces are smooth and correspond to the H landscape having a generic extremum (elliptic contours shrinking to a point) or a generic saddle (hyperbolic contours centred on a cross). On crossing the singularity surface, I has a step discontinuity in the case of an extremum and a logarithmic spike in the case of a saddle.

Of crucial importance in the present study are loci of higher codimension, where the singularity surface is not smooth. Such a locus corresponds to the coalescence of two or more places where ∇H vanishes, i.e. to the *catastrophe set* of degenerate critical points, satisfying

$$\det \begin{Bmatrix} \partial^2 H/\partial x^2 & \partial^2 H/\partial x\, \partial y \\ \partial^2 H/\partial x\, \partial y & \partial^2 H/\partial y^2 \end{Bmatrix} = 0 \tag{8}$$

in addition to (7). As will be described in § 2, the geometrically distinct morphologies of degeneration are classified as catastrophes (Poston and Stewart 1978).

For each catastrophe, I has a power-law divergence as t varies through the catastrophe set on the singularity surface. These divergences dominate the fluctuations of I and contribute to the asymptotic decay of $P(I)$. Higher catastrophes give stronger divergences but are also rarer and so have smaller weight in the ensemble average (6) over $\{C_i\}$. These competing effects give rise to a 'battle of the catastrophes' (§ 3) to determine which degenerate singularity dominates $P(I)$. The remarkable result will be that the battle is won by the elliptic and hyperbolic umbilic catastrophes, giving the universal result

$$P(I) \propto I^{-9}. \tag{9}$$

Section 4 is devoted to a discussion, motivated by example (iv) above, of the very different behaviour of $P(I)$ when the integration in (1) is over one or three dimensions, rather than two. Section 5 is devoted to a discussion, motivated by example (v) above, of the very different behaviour of $P(I)$ when I corresponds to propagation of an initial δ pulse rather than the step of equation (4).

This work makes essential use of catastrophe theory in the form of the extensive classification of singularities developed by Arnold (1973, 1974, 1975), and goes far beyond the now familiar fold, cusp, swallowtail, etc. The central argument will involve scaling laws developed by Berry (1977) and refined for a certain class of cases by Hannay (1982) in a study of strong intensity fluctuations of monochromatic short waves, which are dominated by caustic singularities.

2. Probability-tail exponents

The tail $I \to \infty$ of the ensemble average (6) will be dominated by the Legendre singularity in $\{C_i\}$ space corresponding to the given value of t. It is therefore sensible to expand the integrand, and in particular $I(t, \{C_i\})$ inside the δ function, for $\{C_i\}$

values close to the singularity. The local expansion will be different depending on which catastrophe dominates the Legendre singularity in the $\{C_i\}$ space region under consideration. Let this catastrophe (fold, cusp, umbilic, etc) be labelled j and have codimension K. This is the codimension *within* the Legendre singularity; the codimension in $\{C_i\}$ space is $K+1$, so that a non-degenerate extremum or saddle of H, which is not a catastrophe and so has $K=0$, corresponds to the Legendre singularity with codimension 1, i.e. it is the smooth surface previously discussed. Let the catastrophe be located at $(C_1^*, \ldots, C_K^*)$, so that $C_{i>K}$ correspond to parameter space directions 'along' the singularity and C_0 is an additive constant corresponding to the t value at which the singularity occurs.

Near $\{C_i^*\}$ $(1 \leq i \leq K)$ it is always possible by a diffeomorphism to write H as

$$H = C_0 + \Phi_j(x, y; \{C_i\}) \qquad 1 \leq i \leq K \tag{10}$$

where C_i now denote deviations from C_i^* and where Φ_j is a polynomial normal form (Arnold 1973, 1974, 1975) for the catastrophe j. For example, if j denotes the elliptic umbilic catastrophe,

$$\Phi_{\mathrm{EU}} = x^3 - 3xy^2 + C_3(x^2+y^2) + C_2 y + C_1 x. \tag{11}$$

Therefore (1) becomes, locally,

$$I_j(t - C_0, \{C_i\}) = \iint \mathrm{d}x\, \mathrm{d}y\, \delta(t - C_0 - \Phi_j(x, y; \{C_i\})) \qquad 1 \leq i \leq K. \tag{12}$$

The first objective is to scale away $t - C_0$, to prepare for a further scaling in the ensemble average (6). To this end, (12) is written as a Fourier transform:

$$I_j(t - C_0, \{C_i\}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k}\, \Psi_j(k; \{C_i\}) \exp\{-\mathrm{i}k(t - C_0)\} \tag{12a}$$

where

$$\Psi_j(k; \{C_i\}) \equiv k \iint \mathrm{d}x\, \mathrm{d}y\, \exp\{\mathrm{i}k\Phi_j(x, y; \{C_i\})\}. \tag{13}$$

Trivial relabelling of the k variable gives

$$I_j(t - C_0; \{C_i\}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k}\, \Psi_j\left(\frac{k}{|t - C_0|}; \{C_i\}\right) \exp\{-\mathrm{i}k\, \mathrm{sgn}(t - C_0)\}. \tag{14}$$

Now observe that Ψ_j as defined by (13) is the jth *diffraction catastrophe*, i.e. the monochromatic wavefunction decorating the caustic singularity of the jth catastrophe. As such, it satisfies the scaling law (Berry 1977, 1980, Berry and Upstill 1980)

$$\Psi_j(k; \{C_i\}) = (k/k_0)^{\beta_j}\Psi_j(k_0; \{C_i(k/k_0)^{\sigma_{i,j}}\}) \tag{15}$$

whose origin lies in the fact that each normal form Φ_j (e.g. (11)) can be written as a 'germ', involving x and y but not $\{C_i\}$, plus K 'unfolding terms' linear in $\{C_i\}$. In the waves which (13) describes, β_j describes the divergence of the amplitude $|\Psi_j|$ $(\sim k^{\beta_j})$ at the most singular point on the geometrical caustic as $k \to \infty$, and $\sigma_{i,j}$ describes the shrinking of the fringes in the C_i direction $(\sim k^{-\sigma_{i,j}})$ as $k \to \infty$. An important role is played by

$$\gamma_j \equiv \sum_{i=1}^{K} \sigma_{i,j} \tag{16}$$

which describes the scaling of the K-dimensional hypervolume of the principal diffraction maximum ($\sim k^{-\gamma_j}$) as $k \to \infty$. β_j was introduced by Arnold and studied in detail by Varchenko (1976), and $\sigma_{i,j}$ and γ_j were introduced by Berry (1977).

Inserting the scaling law (15) into (14) and using (12) gives

$$I_j(t-C_0, \{C_i\}) = |t-C_0|^{-\beta_j} I_j(\operatorname{sgn}(t-C_0), \{C_i |t-t_0|^{-\sigma_{i,j}}\}). \tag{17}$$

This shows that as t varies through the contour level C_0 of the singularity, in the case $C_i = 0$ $(1 \leqslant i \leqslant K)$ corresponding to the highest degeneracy of the critical point of H, I has a power-law divergence with exponent β_j (as was shown for the simplest cases by Dangelmayr and Güttinger (1982) in the context of the wave pulse example (v)). For the lowest Legendre singularity, which is not a catastrophe because it corresponds to a non-degenerate critical point of H, β_j is zero and (17) is consistent with the fact, already stated, that the singularity in I is not a power law but a discontinuity or logarithm.

Now the stage is set for evaluating the ensemble average (6). The contribution $P_j(I)$ to the probability distribution $P(I)$, from the jth catastrophe centred on C_i^* $(1 \leqslant i \leqslant K)$, is obtained by transformation to $K+1$ standard coordinates in (10), and use of the scaling (17). This gives

$$P_j(I) = \int dC_{K+1} \ldots \int\int dC_0 \, dC_1 \ldots \int dC_K P_C(C_0, C^*_{1\leqslant i\leqslant K}, C_{i>K}) J(C_0, C^*_{1\leqslant i \leqslant K}, C_{i>K})$$
$$\times \delta[I - |t-C_0|^{-\beta_j} I_j(\operatorname{sgn}(t-C_0), \{C_i|t-C_0|^{-\sigma_{i,j}}\})] \tag{18}$$

where J is a non-singular Jacobian introduced by the transformation. For large I, the δ function restricts C_0 to values near t, and an obvious scaling of C_i $(1 \leqslant i \leqslant K)$ by $|t-C_0|^{\sigma_{i,j}}$ and use of (16) gives

$$P_j(I) = B_j \int dC_1 \ldots \int dC_K \int dC_0 |t-C_0|^{\gamma_j} \delta[I - |t-C_0|^{-\beta_j} I_j(\operatorname{sgn}(t-C_0), \{C_{1\leqslant i \leqslant K}\})] \tag{19}$$

where

$$B_j \equiv \int dC_{K+1} \ldots P_C(C_0 = t, C^*_{1\leqslant i \leqslant K}, C_{i>K}) J(C_0 = t, C^*_{1 \leqslant i \leqslant K}, C_{i>K}). \tag{20}$$

The constants B_j, not involving I, depend on the details of the ensemble of functions H, but the I dependence is embodied in a series of integrals depending only on the normal form for the catastrophe j.

The final step in extracting the I dependence is to integrate over C_0 using the δ function in (19). There are at most (and usually exactly) two contributions from

$$C_{0\pm} = t \mp (I_j(\pm 1, \{C_i\})/I)^{1/\beta_j}. \tag{21}$$

Using standard procedures for integrating a δ function of a function, the final result is obtained as

$$P_j(I) = \frac{B_j}{\beta_j} \left(\int dC_1 \ldots \int dC_K ([I_j(+1, \{C_i\})]^{(\gamma_j+1)/\beta_j} + [I_j(-1, \{C_i\})]^{(\gamma_j+1)/\beta_j}) \right) I^{-\{1+(\gamma_j+1)/\beta_j\}}. \tag{22}$$

Therefore each catastrophe contributes a power-law decay to $P(I)$, with (negative) exponent depending on the detailed algebra of the normal form. The principal contribution obviously comes from the catastrophe giving the smallest exponent, provided this exists, and so

$$P(I) \propto I^{-(1+\mu)} \tag{23}$$

where

$$\mu = \min_{(j)} \left(\frac{\gamma_j + 1}{\beta_j} \right). \tag{24}$$

The computation of μ will be carried out in the next section.

It must be pointed out that the above analysis requires that the two integrals in brackets in (22) converge for the catastrophe j winning the competition (24). Non-rigorous arguments indicating that these integrals do converge are given in appendix 2.

3. Battle of the catastrophes

The computation of the indices β_j and γ_j involved in (24) has been explained by Berry (1977) and will not be repeated here. The results of the computations are embodied in table 1 whose columns will now be explained.

In the first column the singularities are listed using the notation of Arnold (1973, 1974, 1975). Of these, only the cuspoids A_{K+1} have corank 1, meaning that these catastrophes involve critical-point degeneration in just one direction in the xy plane; the remaining singularities have corank 2. It is worth pointing out that the celebrated fold, cusp, swallowtail, butterfly, elliptic/hyperbolic umbilics, and parabolic umbilic catastrophes described by Thom (1975) are, respectively, A_2, A_3, A_4, D_4, D_5.

The second column lists the germs of these catastrophes, that is the normal forms for the function H corresponding to the highest degeneration of critical point. The codimension K, in the third column, is the minimum number of parameters which unfold the degenerate critical point in all possible ways.

In the next column, modality is the number of moduli, denoted by a, b, c in the germs, which label continuous families of catastrophes whose singularities are topologically equivalent but not equivalent under diffeomorphism. In the present context, modality is a technicality not of paramount importance.

In the fifth, sixth and seventh columns the indices β_j and γ_j are listed whose meaning was explained in § 2, and the combination $(\gamma_j + 1)/\beta_j$ occurring in (24). The values of β_j are in full agreement with those calculated previously by Varchenko (1976). The significance of the last column will be made clear in § 4. (While the computation of β_j involves only the germ, the computation of γ_j involves the unfolding terms, listed by Arnold (1974) but not shown in table 1.) (See *Note added in proof.*)

According to (24), the decay exponent μ is the smallest value of $(\gamma_j + 1)/\beta_j$ in the seventh column. For the cuspoids A_{K+1}, the first few values are

$$A_2(\text{fold})\text{: } 10 \qquad A_3(\text{cusp})\text{: } 9 \qquad A_4(\text{swallowtail})\text{: } 9\tfrac{1}{3} \tag{25}$$

and all higher cuspoids have larger values. Therefore the cuspoid competition is won by the *cusp* catastrophe. For the umbilics D_{K+1}, the first few values are

$$D_4(\text{elliptic/hyperbolic})\text{: } 8 \qquad D_5(\text{parabolic})\text{: } 9 \qquad D_6\text{: } 9\tfrac{4}{5} \tag{26}$$

and all higher umbilics have larger values. Therefore the umbilic competition is won

Table 1. Catastrophes of corank 1 and 2.

Catastrophe j	Germ	Codimension K	Modality	β_j	γ_j	$(\gamma_j+1)/\beta_j$	$(\gamma_j+1)/(\beta_j-\frac{1}{2})$
Cuspoid A_{K+1}	$x^{K+2}+y^2$	K	0	$K/[2(K+2)]$	$(K^2+3K)/2(K+2)$	$(K^2+5K+4)/K$	—
Umbilic D_{K+1} $(K\geqslant 3)$	x^2y+y^K	K	0	$(K-1)/2K$	$(K^2+1)/2K$	$(K+1)^2/(K-1)$	—
E_6	x^3+y^4	5	0	$\frac{5}{12}$	$\frac{5}{2}$	$8\frac{2}{5}$	—
E_7	x^3+xy^3	6	0	$\frac{4}{9}$	$\frac{26}{9}$	$8\frac{3}{4}$	—
E_8	x^3+y^5	7	0	$\frac{7}{15}$	$\frac{49}{15}$	$9\frac{1}{7}$	—
X_{j+5} $(j\geqslant 4)$	$x^4+x^2y^2+ay^j$	$j+3$	1	$\frac{1}{2}$	$\frac{7}{2}$	9	—
J_{j+4} $(j\geqslant 6)$	$x^3+x^2y^2+ay^j$	$j+2$	1	$\frac{1}{2}$	4	10	—
Y_{pq} $(5\leqslant p\leqslant q)$	$x^p+x^2y^2+ay^q$	$p+q-1$	1	$\frac{1}{2}$	$(p^2+12)/2p$	$(p^2+12+2p)/p$	—
Z_{11}	$x^3y+y^5+axy^4$	9	1	$\frac{8}{15}$	$\frac{21}{5}$	$9\frac{3}{4}$	156
Z_{12}	$x^3y+xy^4+ax^2y^3$	10	1	$\frac{6}{11}$	$\frac{50}{11}$	$10\frac{1}{6}$	122
Z_{13}	$x^3y+y^6+axy^5$	11	1	$\frac{5}{9}$	$\frac{44}{9}$	$10\frac{3}{5}$	106
W_{12}	$x^4+y^5+ax^2y^3$	10	1	$\frac{11}{20}$	$\frac{9}{2}$	10	110
W_{13}	$x^4+xy^4+ay^6$	11	1	$\frac{9}{16}$	$\frac{77}{16}$	$10\frac{1}{3}$	93
E_{12}	$x^3+y^7+axy^5$	10	1	$\frac{11}{21}$	$\frac{100}{21}$	11	242
E_{13}	$x^3+xy^5+ay^8$	11	1	$\frac{8}{15}$	$\frac{77}{15}$	$11\frac{1}{2}$	184
$Z_{15}=Z_{1,0}$	$x^3y+y^7+ax^2y^3+bxy^6$	12	2	$\frac{4}{7}$	$\frac{39}{7}$	$11\frac{1}{2}$	92
$W_{15}=W_{1,0}$	$x^4+y^6+ax^2y^3+bx^2y^4$	12	2	$\frac{7}{12}$	$\frac{65}{12}$	11	77
$K_{16}=J_{3,0}$	$x^3+ax^2y^3+y^9+bxy^7$	13	2	$\frac{5}{9}$	$\frac{56}{9}$	13	130
N_{16}	$x^4y+ax^3y^2+bx^2y^3+xy^4+cx^3y^3$	12	3	$\frac{3}{5}$	$\frac{28}{5}$	11	66

by the *elliptic/hyperbolic umbilic* catastrophes. For the singularities Y_{pq}, the lowest ($p=5$) has the smallest value, equal to $9\frac{2}{5}$.

It is clear then, from inspection of table 1, that the overall competition is won by D_4 with $\mu=8$, so that (23) indicates that $P(I)$ does indeed have the ninth power decay of equation (9). This is the central result of the present work. Of course, it could happen that some as yet unclassified high catastrophe could have a smaller value of μ, but the trend of table 1 makes this seem most unlikely.

Now consider the special class of systems where the function H in (1) depends on y at most quadratically, i.e. trivially and non-degenerately, so that all essential functional dependence is in the x variation. This happens for examples (ii) and (iii) in § 1 if the Hamiltonian represents a non-relativistic particle moving in a scalar potential, i.e.

$$H=\tfrac{1}{2}y^2+V(x;\{C_i\}). \tag{27}$$

It also happens in example (v) when wave pulses encounter a medium generating a *corrugated* wavefront, i.e. one whose height function h depends on X only. Then only corank-1 catastrophes, namely cuspoids, are permitted to enter the catastrophe competition, and (25) shows that the winning cusp gives a faster decay for $P(I)$, namely

$$P(I)\propto I^{-10}(\text{cuspoids}) \tag{28}$$

as the universal tail for this class of problems.

4. One- and three-dimensional crystal spectra

In the case of crystal spectra (example (iv) of § 1), the restriction to two dimensions is artificial, and it is desirable instead of (1) to study the fluctuations of the more general integral

$$I_n(t,\{C_i\})\equiv\int \mathrm{d}x_1\ldots\int \mathrm{d}x_n\delta(t-H(x_1\ldots x_n;\{C_i\})) \tag{29}$$

especially for $n=1$ and $n=3$.

The analysis proceeds as in § 2, up to equation (12*a*) where instead of $\mathrm{d}k/k$ there appears $\mathrm{d}k/k^{n/2}$. This is because the generalisation of (13) has a prefactor $k^{n/2}$ rather than k—a natural n-dimensional generalisation ensuring that non-degenerate (quadratic) contributions to the phase Φ do not contribute to the caustic singularity as $k\to\infty$. With this definition of the diffraction catastrophes Ψ_j, the scaling law (15) is unchanged, but the important scaling law (17) for I generalises to

$$I_{n,j}(t-C_0,\{C_i\})=|t-C_0|^{-(\beta_j+1-n/2)}I_{n,j}(\operatorname{sgn}(t-C_0),\{C_i|t-t_0|^{\sigma_{i,j}}\}). \tag{30}$$

The rest of the argument proceeds as in § 2, leading to the results (23) and (24) for $P(I_n)$, apart from the substitution

$$\beta_j\to\beta_j+1-n/2. \tag{31}$$

This apparently innocent modification has far-reaching consequences, which will be considered separately for $n=1$ and $n=3$.

For $n = 1$, β_j must be replaced by $\beta_j + \frac{1}{2}$ and only cuspoids (corank 1) may compete. According to (30), even the non-degenerate Legendre singularity ($\beta_j = 0$) gives an inverse square-root divergence in I_1 (a fact obvious from (29) on setting $H \sim x^2$), as opposed to the discontinuities and logarithms when $n = 2$. Moreover, these generic critical points of H give rise to the slowest probability-tail decay, because the first row of table 1 gives

$$(\gamma_j + 1)/(\beta_j + \tfrac{1}{2}) = \tfrac{1}{2}K + 2 \tag{32}$$

which is smallest when $K = 0$. Therefore the fluctuations in one-dimensional crystal spectral densities are not catastrophe dominated, and

$$P(I_1) \propto I_1^{-3}. \tag{33}$$

For $n = 3$, β_j must be replaced by $\beta_j - \frac{1}{2}$, and as well as catastrophes in table 1 it is necessary to allow catastrophes of corank 3 to compete. But now, according to (30), no catastrophe with $\beta_j \leq \frac{1}{2}$ gives rise to a power-law divergence in I_3. In particular, generic critical points ($\beta_j = 0$) give singularities of the type $(t - C_0)^{+1/2}$ in I_3. All catastrophes with $\beta_j \leq \frac{1}{2}$—which include all cuspoids A_{K+1} and umbilics D_{K+1}—can therefore be ignored in the comparison leading to the probability decay exponent μ.

The remaining catastrophes include all those in table 1 below and including Z_{11}, and the ten corank-3 catastrophes in table 2. The competing values of $(\gamma_j + 1)/(\beta_j - \frac{1}{2})$ are listed in the last columns of tables 1 and 2. For the lowest competing catastrophe, Q_{10} with $K = 8$, the decay from (23) is I_3^{-115}. Unfortunately, the most complicated of all the listed catastrophes—V_{15} with three moduli and corank 3—is the one giving the smallest exponent, i.e. 48. It therefore seems that the present classification of singularities of corank 2 and corank 3 is not sufficiently extensive to include the winner of this competition. All that can be said at this stage is that

$$P(I_3) \geq I_3^{-48} \qquad \text{as } I_3 \to \infty \tag{34}$$

or, more precisely,

$$\lim_{I_3 \to \infty} \frac{\mathrm{d}\ln(P(I_3))}{\mathrm{d}\ln(I_3^{-1})} \leq 48. \tag{35}$$

Table 2. Some corank-3 catastrophes with $\beta_j > \frac{1}{2}$.

Catastrophe j	Germ	Codimension K	β_j	γ_j	$(\gamma_j + 1)/(\beta_j - \frac{1}{2})$
Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	8	$\frac{13}{24}$	$\frac{15}{4}$	114
Q_{11}	$x^3 + y^2z + xz^3 + az^5$	9	$\frac{5}{9}$	4	90
Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	10	$\frac{17}{30}$	$\frac{13}{3}$	80
S_{11}	$x^4 + y^2z + xz^2 + ax^3z$	9	$\frac{9}{16}$	$\frac{63}{16}$	79
S_{12}	$x^2y + y^2z + xz^3 + az^5$	10	$\frac{15}{26}$	$\frac{55}{13}$	68
U_{12}	$x^3 + y^3 + z^4 + zxyz^2$	10	$\frac{7}{12}$	$\frac{25}{6}$	62
$Q_{14} = Q_{2,0}$	$x^3 + y^6 + yz^2 + ax^2y^2 + by^7$	11	$\frac{7}{12}$	5	72
$S_{14} = S_{1,0}$	$x^2y + xz^2 + y^5 + axy^3 + bxy^4$	11	$\frac{3}{5}$	$\frac{24}{5}$	58
$U_{14} = U_{1,0}$	$x^3 + y^3 + xz^3 + ayz^3 + bzy^4$	11	$\frac{11}{18}$	$\frac{14}{3}$	51
$V_{15} = V_{1,0}$	$x^4 + y^4 + xz^2 + ayz^2 + bx^2y^2 + cy^2z^2$	11	$\frac{5}{8}$	$\frac{39}{8}$	47

5. Fluctuations in δ-wave pulses

In the case of example (v) of § 1, it is natural to consider not only the propagation of an initial step discontinuity (4) but also the propagation of an initial δ pulse, i.e. the time derivative of (4). The resulting wave (which may now be negative) is simply the time derivative of (5), and its magnitude can be expressed in terms of (1) by the positive quantity

$$I'(t, \{C_i\}) \equiv \left|\frac{\partial}{\partial t} I(t; \{C_i\})\right|. \tag{36}$$

In order to find the probability tail for the fluctuations in I', the argument of § 2 is employed, with a slight modification of (12) and (14) leading to the replacement of equation (17) by

$$I'_j(t - C_0, \{C_i\}) = |t - C_0|^{-(\beta_j+1)} I'_j(\mathrm{sgn}(t - C_0), \{C_i |t - t_0|^{\sigma_{i,j}}\}). \tag{37}$$

The rest of the argument proceeds unchanged, and leads to the results (23) and (24) for $P(I')$, apart from the substitution

$$\beta_j \to \beta_j + 1 \tag{38}$$

(which, curiously, is formally identical with (31) for zero dimensionality n).

Now the competition includes all the catastrophes of table 1. But it is won by none of them! The smallest value of $(\gamma_j+1)/(\beta_j+1)$ occurs for $\beta_j = \gamma_j = 0$, corresponding not to any catastrophe but to generic saddles in H and giving the powerful tail

$$P(I') \propto (I')^{-2}. \tag{39}$$

This result can be made plausible by the following elementary argument. A generic saddle contributes a logarithmic divergence to I, so that I' diverges like

$$I' \propto |t - C_0|^{-1}. \tag{40}$$

Therefore

$$P(I') \propto \int \mathrm{d}C_0\, \delta(I' - |t - C_0|^{-1}) \propto (I')^{-2}. \tag{41}$$

But this argument does not establish the fact that no higher singularity can give a stronger probability tail—only the foregoing analysis, together with table 1, can demonstrate the lack of catastrophe dominance.

6. Summary

The statistics of I, defined by equation (1), are indeed remarkable. If H is an unrestricted random function of x and y, the main result (9) holds, that $P(I) \propto I^{-9}$. This is universal, i.e. independent of the detailed ensemble of H. But a slight change in the mathematical framework can shift the statistics into a different universality class, with a different decay law. For example, restricting H to be essentially one-dimensional (i.e. at most quadratic in y) gave the law (28), namely $P(I) \propto I^{-10}$. On the other hand, (33) shows that making the integral (1) one-dimensional changes the decay to $P(I_1) \propto I_1^{-3}$ whilst (34) shows that three-dimensional integrals have probability distributions decaying no faster than $P(I_3) \propto I_3^{-48}$. Finally, the absolute value of the derivative of I decays according to (39) as $P(I') \propto (I')^{-2}$.

Appendix 1. Pulse diffraction theory

The first step in deriving (5) is to write the boundary condition (4) as a treble Fourier transform over variables k, conjugate to ct, and $\boldsymbol{Q} \equiv (Q_x, Q_y)$, conjugate to $\boldsymbol{R} = (X, Y)$. This gives

$$\phi(\boldsymbol{R}, 0, t) = -\frac{1}{(2\pi)^3 \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k + \mathrm{i}\varepsilon} \exp(-\mathrm{i}kct) \iint \mathrm{d}\boldsymbol{r} \times \exp(-\mathrm{i}kh(\boldsymbol{r})) \iint \mathrm{d}\boldsymbol{Q} \exp\{\mathrm{i}\boldsymbol{Q} \cdot (\boldsymbol{R} - \boldsymbol{r})\} \tag{A1}$$

where ε is a positive infinitesimal and $\boldsymbol{r} \equiv (x, y)$. Now we use the fact that the solution of the wave equation, which when $Z = 0$ has the form

$$\exp\{\mathrm{i}(\boldsymbol{Q} \cdot \boldsymbol{R} - kct)\} \tag{A2}$$

and which propagates towards $Z = +\infty$, is the plane wave

$$\exp\{\mathrm{i}(\boldsymbol{Q} \cdot \boldsymbol{R} + (k^2 - Q^2)^{1/2} Z - kct)\} \tag{A3}$$

where the square root is positive imaginary if $|Q| > k$, corresponding to a wave decaying for increasing Z. From (A1), the required superposition of these plane waves is

$$\phi(\boldsymbol{R}, Z, t) = -\frac{1}{(2\pi)^3 \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k + \mathrm{i}\varepsilon} \exp(-\mathrm{i}kct) \iint \mathrm{d}\boldsymbol{r} \times \exp(-\mathrm{i}kh(\boldsymbol{r})) \iint \mathrm{d}\boldsymbol{Q} \exp \mathrm{i}\{\boldsymbol{Q} \cdot (\boldsymbol{R} - \boldsymbol{r}) + (k^2 - Q^2)^{1/2} Z\}. \tag{A4}$$

Evaluating the angular and radial part of the $\boldsymbol{Q}$ integral gives

$$\iint \mathrm{d}\boldsymbol{Q} \exp\{\mathrm{i}(\boldsymbol{Q} \cdot (\boldsymbol{R} - \boldsymbol{r}) + (k^2 - Q^2)^{1/2} Z)\} = 2\pi \int_0^{\infty} \mathrm{d}Q\, Q J_0(Q|\boldsymbol{R} - \boldsymbol{r}|) \exp\{\mathrm{i}(k^2 - Q^2)^{1/2} Z\} = \frac{Z}{[Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{3/2}} \{1 - \mathrm{i}k[Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{1/2}\} \exp\{\mathrm{i}k[Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{1/2}\}. \tag{A5}$$

Inserting this into (A4) and performing the k integration gives the following exact representation of the wavefunction:

$$\phi(\boldsymbol{R}, Z, t) = \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{r} \left(\frac{Z}{[Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{3/2}} \Theta(ct + h(\boldsymbol{r}) - [Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{1/2}) + \frac{Z}{Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2} \delta(ct + h(\boldsymbol{r}) - [Z^2 + (\boldsymbol{R} - \boldsymbol{r})^2]^{1/2}) \right). \tag{A6}$$

The desired paraxial approximation (5) follows immediately on expanding for $|\boldsymbol{R}-\boldsymbol{r}|/Z$ small and neglecting the first term which is of higher order in Z^{-1}. These procedures are justified by an argument based on the fact that ϕ is dominated by its singularities (7), taken in conjunction with the assumed small slopes of the wavefront deformation h.

It is worth noting that in the exact wave (A6) the boundary condition (4) emerges from the first term as $Z\to 0$, and this term is neglected in the approximation (5); but in spite of this (5) reduces *exactly* to (4) as $Z\to 0$.

Appendix 2. Convergence of integrals

The coefficients in (22), which must be finite in order that the power law (23) correctly describes the asymptotics of $P(I)$, depend on the integrals

$$J_j^{\pm} \equiv \int \mathrm{d}C_1 \ldots \int \mathrm{d}C_K \{I_j(\pm 1, \{C_i\})\}^{(\gamma_j+1)/\beta_j} \tag{A7}$$

where the I_j are given by (12) with $t-C_0=\pm 1$. According to § 3 the catastrophe competition was won by (a) the cusp (for the corank-1 case) and (b) the elliptic/hyperbolic umbilic (for the unrestricted corank-2 case); therefore the convergence of $J_j^{\pm}$ need be studied for only these two catastrophes. In each case the 'dangerous' regions of $\{C_i\}$ space, where the integral is liable to diverge, are (i) close to the Legendre singularity defined by (7) and (10), and (ii) $|C_i|\to\infty$. To save tedious and inessential complication in what follows, numerical factors will be set equal to unity, variables will be scaled without relabelling, and the operation Re will be implied to make all square roots vanish whenever their argument is negative.

(a) For the *cusp*, (A7) and the results of § 3 give

$$\begin{aligned} J_{\text{cusp}} &= \int \mathrm{d}C_1 \int \mathrm{d}C_2 \bigg(\int \mathrm{d}x \int \mathrm{d}y\, \delta(\pm 1 - y^2 - x^4 - C_2 x^2 - C_1 x) \bigg)^9 \\ &= \int \mathrm{d}C_1 \int \mathrm{d}C_2 \bigg(\int \mathrm{d}x (\pm 1 - x^4 - C_2 x^2 - C_1 x)^{-1/2} \bigg)^9 . \end{aligned} \tag{A8}$$

(i) In the full space of t and $\{C_i\}$, the Legendre singularity for the cusp catastrophe (Arnold 1976, Sewell 1977) consists of a swallowtail surface with its highest singularity at the origin. The sections $t=\pm 1$ involved in (A8) miss the origin, and in the plane the singularity is a curve which in one of the cases $\pm$ has two cusps. To examine convergence near one of these cusps it is convenient to make a diffeomorphism to local normal form in terms of new variables x, y and parameters C_1, C_2 with the cusp at the origin, i.e.

$$\pm 1 - x^4 - C_2 x^2 - C_1 x \to C_2 - x^3 - C_1 x \tag{A9}$$

(the transformation can be carried out explicitly but the intricate details are irrelevant here). Therefore it is necessary to study the convergence for small C_1, C_2 of

$$J = \int \mathrm{d}C_1 \int \mathrm{d}C_2 \bigg(\int \mathrm{d}x (C_2 - x^3 - C_1 x)^{-1/2} \bigg)^9 . \tag{A10}$$

The transformations $x/C_2^{1/3} \to x$ and $C_1/C_2^{2/3} \to C_1$ give, successively,

$$J = \int \mathrm{d}C_2\, C_2^{-3/2} \int \mathrm{d}C_1 \Big(\int \mathrm{d}x (1 - x^3 - C_1 C_2^{-2/3} x)^{-1/2} \Big)^9$$
$$= \int \mathrm{d}C_2\, C_2^{-5/6} \int \mathrm{d}C_1 \Big(\int \mathrm{d}x (1 - x^3 - C_1 x)^{-1/2} \Big)^9. \qquad \text{(A11)}$$

The singularity at $C_2 = 0$ is integrable and there is no singularity at $C_1 = 0$. Therefore J_{cusp} converges near the Legendre singularity.

(ii) In the asymptotic regime $|C_1| \to \infty$, $|C_2| \to \infty$, the transformation $x/C_2^{1/2} \to x$ turns (A8) into

$$J^{\pm}_{\text{cusp}} = \int \mathrm{d}C_1 \int \mathrm{d}C_2\, C_2^{-9/2} \Big(\int \mathrm{d}x (-x^4 - x^2 - C_1 C_2^{-3/2} x \pm C_2^{-2})^{-1/2} \Big)^9. \qquad \text{(A12)}$$

The term $\pm C_2^{-2}$ can be neglected for large $|C_2|$, and the successive transformations $C_1 C_2^{-3/2} \to C_1$ and $x C_1^{-1/3} \to x$ give

$$J_{\text{cusp}} = \int \mathrm{d}C_1 \int \mathrm{d}C_2\, C_2^{-3} \Big(\int \mathrm{d}x (-x^4 - x^2 - C_1 x)^{-1/2} \Big)^9$$
$$= \int \mathrm{d}C_1\, C_1^{-3} \int \mathrm{d}C_2\, C_2^{-3} \Big(\int \mathrm{d}x (-x^4 - x - x^2 C_1^{-2/3})^{-1/2} \Big)^9. \qquad \text{(A13)}$$

For large $|C_1|$ the last term can be neglected leaving a convergent integral over x, and the C_1 and C_2 integrals converge at infinity. This concludes the discussion of J_{cusp}.

(*b*) For the *elliptic/hyperbolic umbilic*, (A7) and the results of § 3, together with the fact that $J^+ = J^- \equiv J$, give

$$J_{\text{umbilic}} = \int \mathrm{d}C_1 \int \mathrm{d}C_2 \int \mathrm{d}C_3 \Big(\int \mathrm{d}x \int \mathrm{d}y\, \delta(1 - x^3 - 3\varepsilon x y^2 - C_3(x^2 + y^2) - C_1 x - C_2 y) \Big)^8 \qquad \varepsilon = +1 \text{ (hyperbolic)}; -1 \text{ (elliptic)}. \qquad \text{(A14)}$$

(i) The section $t = 1$ misses the highest singularity of the Legendre singularity in t, $\{C_i\}$ space. Within this section, i.e. in C_1, C_2, C_3 space, the highest singularities are swallowtail surfaces (three in the elliptic case, one in the hyperbolic case (see Arnold 1976, Sewell 1978)). To examine convergence near one of these swallowtails it is convenient to make a diffeomorphism to local normal form in terms of new variables x, y and parameters C_1, C_2, C_3 with the swallowtail at the origin, i.e.

$$1 - x^3 - C_3(x^2 + y^2) - C_1 x - C_2 y \to C_3 - y^2 - x^4 - C_2 x^2 - C_1 x. \qquad \text{(A15)}$$

Therefore it is necessary to study the convergence for small C_1, C_2, C_3 of

$$J = \int \mathrm{d}C_1 \int \mathrm{d}C_2 \int \mathrm{d}C_3 \Big(\int \mathrm{d}x (C_3 - x^4 - C_2 x^2 - C_1 x)^{-1/2} \Big)^8. \qquad \text{(A16)}$$

The successive transformations $x C_3^{-1/4} \to x$ and $C_2 C_3^{-1/2} \to C_2$, $C_1 C_3^{-3/4} \to C_1$ give

$$J = \int \mathrm{d}C_3\, C_3^{-3/4} \int \mathrm{d}C_1 \int \mathrm{d}C_2 \Big(\int \mathrm{d}x (1 - x^4 - C_2 x^2 - C_1 x)^{-1/2} \Big)^8. \qquad \text{(A17)}$$

This closely resembles J^+_{cusp} (A8) and can be simplified by an argument exactly analogous to that leading to (A11), with the result

$$J=\int \mathrm{d}C_3\, C_3^{-3/4}\int \mathrm{d}C_2\, C_2^{-2/3}\int \mathrm{d}C_1\left(\int \mathrm{d}x(1-x^3-C_1x)^{-1/2}\right)^8. \quad \text{(A18)}$$

The singularities at $C_2=C_3=0$ are integrable and there is no singularity at $C_1=0$. Therefore J_{umbilic} converges near the Legendre singularity.

(ii) In the asymptotic regime $|C_1|\to\infty$, $|C_2|\to\infty$, $|C_3|\to\infty$, the transformations $x/C_3\to x$, $y/C_3\to y$ followed by $C_1/C_3^2\to C_1$, $C_2/C_3^2\to C_2$ turn (A14) into

$$J_{\mathrm{umbilic}}=\int \mathrm{d}C_3\, C_3^{-4}\int \mathrm{d}C_1\int \mathrm{d}C_2\left(\int \mathrm{d}x\int \mathrm{d}y\,\delta(x^3+3\varepsilon x^2y + x^2+y^2+C_1x+C_2y-C_3^{-3})\right)^8. \quad \text{(A19)}$$

The term C_3^{-3} can be neglected for large $|C_3|$, and the introduction of polar coordinates Q, θ in the C_1, C_2 plane, followed by polar coordinates $Q^{1/2}r$, ϕ in the x, y plane, gives

$$J_{\mathrm{umbilic}}=\int \mathrm{d}C_3\, C_3^{-4}\int \mathrm{d}Q\, Q^{-3}\int_0^{2\pi}\mathrm{d}\theta\left(\int_0^\infty \mathrm{d}r\, r\int_0^{2\pi}\mathrm{d}\phi\,\delta[r^3(\cos^3\phi+3\varepsilon\cos\phi\sin^2\phi) + r\cos(\theta-\phi)+r^2Q^{-1/2}]\right)^8. \quad \text{(A20)}$$

The term $r^2Q^{-1/2}$ can be neglected for large Q. Evidently the integrals over C_3 and Q converge at infinity, leaving the convergence of J_{umbilic} dependent on the θ integral, which after performing the r integration with the aid of the δ function becomes J_θ where

$$J_\theta\equiv\int_0^{2\pi}\mathrm{d}\theta\left(\int_0^{2\pi}\mathrm{d}\phi[-(\cos^3\phi+3\varepsilon\cos\phi\sin^2\phi)\cos(\theta-\phi)]^{-1/2}\right)^8. \quad \text{(A21)}$$

Singularities occur for those θ values for which the ϕ integrand contains ϕ values where both trigonometric factors in the square root vanish. For the elliptic and hyperbolic umbilics there are, respectively, three and one such θ, corresponding to the asymptotic lines of self-intersection of sheets of the Legendre singularity in C_1, C_2, C_3 space as depicted by Arnold (1976) and Sewell (1978). However these singularities in the θ integrand are of the type $\ln^8\theta$ and hence integrable. This concludes the discussion of J_{umbilic}.

Note added in proof. Chillingworth and Romero (private communication) have obtained the remarkable relation $\gamma_j+1=(K+1)(1-\beta_j)$ between the catastrophe exponents. This holds for all non-modal catastrophes, and also for some modal catastrophes provided the modal terms are considered to belong not to the germ but to the unfolding, where they increase K by one and contribute negatively to γ_j (β_j is unaffected). In the present context such negative weights appear unphysical and this reassignment of modal terms does not seem to affect any conclusions concerning catastrophe dominance.

References

Arnold V I 1973 *Usp. Mat. Nauk* **28** No 5 17–44 (1973 *Russ. Math. Surv.* **28** No 5 19–48)
—— 1974 *Usp. Mat. Nauk* **29** (No 2) 11–49 (1974 *Russ. Math. Surv.* **29** (No 2) 10–50)
—— 1975 *Usp. Mat. Nauk* **30** (No 5) 3–65 (1975 *Russ. Math. Surv.* **30** (No 5) 1–75)

—— 1976 *Comm. Pure. Appl. Maths* **29** 557–82
—— 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)
Berry M V 1972 *J. Phys. A: Gen. Phys.* **5** 272–91
—— 1977 *J. Phys. A: Math. Gen.* **10** 2061–81
—— 1980 *Proc. Symp. Pure Appl. Math* **36** 13–28
Berry M V and Mackley M R 1977 *Phil. Trans. R. Soc.* A **287** 1–16
Berry M V and Upstill C 1980 *Prog. Opt.* **18** 257–346
Dangelmayr G and Güttinger W 1982 *Geophys. J. R. Astron. Soc.* in press
Hannay J H 1982 *Opt. Acta* submitted
Poston T and Stewart I N 1978 *Catastrophe Theory and its Applications* (London: Pitman)
Sewell M J 1977 *Math. Proc. Camb. Phil. Soc.* **82** 147–63
—— 1978 *Math. Proc. Camb. Phil. Soc.* **83** 273–88
Thom R 1975 *Structural Stability and Morphogenesis* (Reading, Mass: Benjamin)
Varchenko A N 1976 *Funkt. An.* **10** 13–38
Wannier G H 1959 *Elements of Solid State Theory* (Cambridge: Cambridge University Press)